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# A class of $r$-semipreinvex functions and optimality in nonlinear programming

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**Abstract** In this paper, a class of functions, named as $r$-semipreinvex functions, which is generalization of semipreinvex functions and $r$-preinvex functions, is introduced. Example is given to show that there exists function which is $r$-semipreinvex function, but is not semipreinvex function. Furthermore, some basic characterizations of $r$-semipreinvex functions are established. At the same time, some optimality results are obtained in nonlinear programming problems.

**Keywords** Semi-connected sets · Semipreinvex functions · $r$-Semipreinvex functions · Optimality · Nonlinear programming

**Mathematics Subject Classification (2000)** 90C25 · 90C26 · 90C30

## 1 Introduction

It is well known that convexity has been playing a key role in mathematical programming, engineering and optimization theory. The research on characterizations and generalizations of convexity is one of the most important aspects in mathematical programming and optimization theory in [1,2]. Hanson introduced the concept of invex functions which is extension of differentiable convex functions and proved the sufficiency of Kuhn–Tucker condition in [3]. For one more general case, Ben-Israel and Mond [4] considered functions (not necessarily differentiable) for which there exists a vector function $\eta : R^n \times R^n \to R^n$ such that, for any $x, y \in R^n$, $\lambda \in [0, 1]$,

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$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda) f(y). \tag{1}$$

Weir et al. named such kinds of functions satisfying the condition (1) as preinvex functions with respect to vector-valued function $\eta$ in [5,6]. Invex functions and preinvex functions are two important kinds of generalized convexity. Many authors discussed some properties and the applications in mathematical programming about invexity and preinvexity in the literature [7–16].

On the other hand, Avriel introduced the definition of $r$-convex functions which is another generalization of convex functions, discussed some characterizations and its relations with other generalizations of convex functions in [16]. In [17], Antczak introduced the concept of a class of $r$-preinvex functions which is a generalization of $r$-convex functions and preinvex functions and obtained some optimality results under appropriate $r$-preinvexity conditions for constrained optimization problems. Further study on characterizations and extensions of $r$-preinvex functions has been done in [18–21].

Moreover, Antczak introduced the concept of $G$-preinvex functions and obtained some optimality results under $G$-preinvexity conditions for constrained optimization problems in [22]. As a generalization of preinvex functions, Yang et al. established the definition of a class of semipreinvex functions and discussed the applications in prevariational inequality in [23]. Furthermore, Yang et al. discussed some properties and saddle point optimality criteria for multiobjective fractional programming problem under semipreinvexity conditions in [24]. As a generalization of $B$-vex functions and semipreinvex functions, Long and Peng introduced the concept of semi-$B$-preinvex functions and obtained some optimality results in [25]. Furthermore, Yuan and Chinchuluun et al. studied generalized convexity and the applications in multiobjective programming in the literature [26–29].

Motivated by works of [17,24,25] and the references therein, in this paper, we propose the concept of $r$-semipreinvex functions and obtain some important characterizations and optimality results in nonlinear programming. The concept $r$-semipreinvexity unifies the concepts of $r$-preinvexity and semipreinvexity.

## 2 Preliminaries and definitions

Avriel [16] introduced the concept of a class of $r$-convex functions as follows:

**Definition 2.1** [16] Let $f : S \to R$, where $S$ is a nonempty convex set in $R^n$. $f$ is said to be *r-convex* on $S$ if, for any $x, y \in S, \lambda \in [0, 1]$,

$$f(y + \lambda(x - y)) \leq \begin{cases} \log\left(\lambda e^{rf(x)} + (1 - \lambda)e^{rf(y)}\right)^{\frac{1}{r}}, & r \neq 0, \\ \lambda f(x) + (1 - \lambda) f(y), & r = 0. \end{cases}$$

*Remark 2.1* Any convex function is an $r$-convex function with $r = 0$. But the converse is not necessarily true.

*Example 2.1* Let $S = [-1, 1]$, $f : S \to R$ be defined by $f(x) = \log(1 + |x|)$.

Then, it is easy to check that $f$ is an 1-convex function. But $f(x)$ is not a convex function on $S$, for $f(y + \lambda(x - y)) > \lambda f(x) + (1 - \lambda) f(y)$ holds when $x = \frac{1}{4}, y = \frac{3}{4}, \lambda = \frac{1}{2}$.

Weir and Mond [5], Weir and Jeyakumar [6] presented the concepts of invex sets and preinvex functions, respectively, as follows:

**Definition 2.2** [5,6] A set $S \subseteq R^n$ is said to be *invex* if there exists $\eta : S \times S \to R^n$ such that, for any $x, y \in S, \lambda \in [0, 1], y + \lambda\eta(x, y) \in S$.

**Definition 2.3** [5,6] Let $S \subseteq R^n$ be invex with respect to $\eta : S \times S \to R^n$. $f : S \to R$ is said to be *preinvex* with respect to the same $\eta$ if, for any $x, y \in S, \lambda \in [0, 1]$, $f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda) f(y)$.

*Remark 2.2* Any convex set and any convex function are an invex set and a preinvex function with $\eta(x, y) = x - y$, respectively. But the converse is not necessarily true.

*Example 2.2* Let $S = [-1, 1]$. Obviously, $S$ is an invex set with respect to $\eta(x, y)$, where

$$\eta(x, y) = \begin{cases} x - y, & -1 \leq x < 0, -1 \leq y < 0, \\ x - y, & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ -x - y, & 0 \leq x \leq 1, -1 \leq y < 0, \\ -y - x^2 - 2x, & -1 \leq x < 0, 0 \leq y \leq 1. \end{cases}$$

Let $f : S \to R$ be defined by $f(x) = -|x|$.

Then, we can verify that $f$ is preinvex function with respect to $\eta$. But by the fact that

$$f\,(y + \lambda(x - y)) > \lambda f(x) + (1 - \lambda) f(y)$$

holds when $x = -\frac{1}{2}, y = \frac{1}{2}, \lambda = \frac{1}{2}$, we see that $f$ is not convex on $S$.

**Definition 2.4** [17] Let $S \subseteq R^n$ be invex with respect to $\eta : S \times S \to R^n$. $f : S \to R$ is said to be *r-preinvex* if, for any $x, y \in S, \lambda \in [0, 1]$,

$$f(y + \lambda\eta(x, y)) \leq \begin{cases} \log\left(\lambda e^{rf(x)} + (1 - \lambda)e^{rf(y)}\right)^{\frac{1}{r}}, & r \neq 0, \\ \lambda f(x) + (1 - \lambda) f(y), & r = 0. \end{cases}$$

*Remark 2.3* The preinvex function is a $r$-preinvex function with $r = 0$, $r$-convex function is $r$-preinvex function with $\eta(x, y) = x - y$. But the converse is not necessarily true.

*Example 2.3* Let $S = [-1, 1]$. Then $S$ is an invex set with respect to $\eta(x, y)$, where

$$\eta(x, y) = \begin{cases} x - y, & -1 \leq x < 0, -1 \leq y < 0, \\ -y + 2x - x^2, & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ -x - y, & 0 \leq x \leq 1, -1 \leq y < 0, \\ -y - x^2 - 2x, & -1 \leq x < 0, 0 \leq y \leq 1. \end{cases}$$

Let $f : S \to R$ be defined by $f(x) = \log(2 - |x|)$.

Then, we can verify that $f$ is 1-preinvex function with respect to $\eta$. Nevertheless $f$ is not preinvex function with respect to $\eta$ on $S$, due to the fact that

$$f(y + \lambda\eta(x, y)) > \lambda f(x) + (1 - \lambda) f(y)$$

holds for $x = 0, y = 1, \lambda = \frac{1}{2}$ and $f$ is not 1-convex function on $S$, for

$$f\,(y + \lambda\eta(x, y)) > \log\left(\lambda e^{f(x)} + (1 - \lambda)e^{f(y)}\right)$$

holds when $x = 1, y = -1, \lambda = \frac{1}{2}$.

**Definition 2.5** [23] $S \subseteq R^n$ is said to be *semi-connected set* if there exists $\eta : S \times S \times [0, 1] \to R^n$ such that, for any $x, y \in S, \lambda \in [0, 1]$, $y + \lambda\eta(x, y, \lambda) \in S$.

**Definition 2.6** [23] Let $S \subseteq R^n$ be semi-connected set with respect to $\eta : S \times S \times [0,1] \to R^n$. $f : S \to R$ is said to be *semipreinvex* with respect to $\eta$ if, for any $x, y \in S, \lambda \in [0,1]$, $\lim_{\lambda \to 0} \lambda\eta(x,y,\lambda) = 0$, $f(y+\lambda\eta(x,y,\lambda)) \le \lambda f(x) + (1-\lambda) f(y)$.

**Definition 2.7** Let $S \subseteq R^n$ be semi-connected set with respect to $\eta : S \times S \times [0,1] \to R^n$. $f : S \to R$ is said to be *r-semipreinvex* with respect to $\eta$ if, for any $x, y \in S, \lambda \in [0,1]$, $\lim_{\lambda \to 0} \lambda\eta(x,y,\lambda) = 0$ and

$$f(y+\lambda\eta(x,y,\lambda)) \le \begin{cases} \log\left(\lambda e^{rf(x)} + (1-\lambda)e^{rf(y)}\right)^{\frac{1}{r}}, & r \neq 0, \\ \lambda f(x) + (1-\lambda) f(y), & r = 0. \end{cases} \tag{2}$$

The term of *r-semipreincave* is defined in a similar way with the sense of the inequality reversed. $f : S \to R$ is said to be *strictly r-semipreinvex* with respect to $\eta$, if the inequality (2) is strict, for all $x, y \in S, x \neq y, \lambda \in (0,1)$.

*Remark 2.4* Every semipreinvex function with respect to $\eta$ is $r$-semipreinvex function with respect to the same $\eta$, where $r = 0$. However, the converse is not true.

*Example 2.4* Let $S = [-5, 5]$, it is easy to check that $S$ is a semi-connected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda \to 0} \lambda\eta(x,y,\lambda) = 0$, where

$$\eta(x,y,\lambda) = \begin{cases} \frac{x-y}{\sqrt{\lambda}}, & -5 \le x < 0, -5 \le y < 0, x > y, 0 < \lambda \le 1, \\ \lambda(x-y), & 0 \le x \le 5, 0 \le y \le 5, x \ge y, \\ \lambda(x-y), & -5 \le x < 0, -5 \le y < 0, x \le y, \\ x-y, & 0 \le x \le 5, 0 \le y \le 5, x < y, \\ x-y, & 0 \le x \le 5, -5 \le y < 0, x < -y, \\ x-y, & -5 \le x < 0, 0 \le y \le 5, x > -y, \\ 0, & 0 \le x \le 5, -5 \le y < 0, x \ge -y, \\ 0, & -5 \le x < 0, 0 \le y \le 5, x \le -y. \end{cases}$$

Let $f : S \to R$ be defined by $f(x) = \log(x^2 + 1)$.

Then, we can verify that $f$ is 1-semipreinvex function with respect to $\eta$. But $f$ is not a semipreinvex function with respect to the same $\eta$, for

$$f(y+\lambda\eta(x,y,\lambda)) > \lambda f(x) + (1-\lambda) f(y)$$

when $x = 2, y = 4, \lambda = \frac{1}{2}$.

*Remark 2.5* Every $r$-convex function is $r$-semipreinvex function with respect to $\eta(x,y,\lambda) = x - y$. But the converse is not true.

*Example 2.5* Let $S = (-5, 5)$, it is easy to check that $S$ is a semi-connected set with respect to $\eta(x,y,\lambda)$ and $\lim_{\lambda \to 0} \lambda\eta(x,y,\lambda) = 0$, where

$$\eta(x,y,\lambda) = \begin{cases} \lambda(x-y), & 0 \le x < 5, 0 \le y < 5, x < y, \\ \lambda(x-y), & -5 < x < 0, -5 < y < 0, x > y, \\ \frac{x-y}{\sqrt{\lambda}}, & 0 \le x < 5, 0 \le y < 5, x \ge y, 0 < \lambda \le 1, \\ \frac{x-y}{\sqrt{\lambda}}, & -5 < x < 0, -5 < y < 0, x \le y, 0 < \lambda \le 1, \\ -x-y, & 0 \le x < 5, -5 < y < 0, x \ge -y, \\ -x-y, & -5 < x < 0, 0 \le y < 5, x \le -y, \\ 0, & 0 \le x < 5, -5 < y < 0, x < -y, \\ 0, & -5 < x < 0, 0 \le y < 5, x > -y. \end{cases}$$

Let $f : S \to R$ be defined by $f(x) = \log(5- \mid x \mid)$.

Then we can verify that $f$ is a 1-semipreinvex function with respect to $\eta$. But $f$ is not a 1-convex function, for

$$f(y + \lambda(x - y)) > \log\left(\lambda e^{rf(x)} + (1 - \lambda)e^{rf(y)}\right)^{\frac{1}{r}}$$

when $x = 1$, $y = -1$, $\lambda = \frac{1}{2}$.

**Lemma 2.1** [24] *Let $I$ be an index set. $(S_i)_{i \in I}$ is a family of semi-connected subsets in $R^{n+1}$ with respect to the same function $\eta' : R^{n+1} \times R^{n+1} \times [0, 1] \to R^{n+1}$, then their intersection $\bigcap_{i \in I} S_i$ is a semi-connected set with respect to the same $\eta'$.*

## 3 Some properties of *r*-semipreinvex functions

We give a few results concerning some basic properties of $r$-semipreinvexity. Since the proofs of these results are obvious, they have been omitted in this paper.

**Theorem 3.1** *The following statements hold.*

(a) *Let $f$ be $r$-semipreinvex ($r$-semipreincave) function with respect to $\eta$ on $S \subset R^n$, and $\alpha$ any real number. Then the function $f + \alpha$ is $r$-semipreinvex ($r$-semipreincave) with respect to $\eta$ on $S$.*
(b) *Let $f$ be $r$-semipreinvex ($r$-semipreincave) function with respect to $\eta$ on $S \subset R^n$, and $k$ any positive real number. Then $kf$ is $\frac{r}{k}$-semipreinvex ($\frac{r}{k}$-semipreincave) function with respect to $\eta$ on $R^n$.*
(c) *$f : S \subset R^n \to R$ is $r$-semipreinvex function with respect to $\eta$ if and only if $-f$ is $(-r)$-semipreincave function with respect to $\eta$.*
(d) *Let $f$ be a function with real values, defined on $S \subset R^n$, $g$ be a function defined by $g(x) := e^{rf(x)}$, where $r$ is any real number. Then $f$ is $r$-semipreinvex ($r$-semipreincave) function with respect to $\eta$ if and only if the function $g$ is $r$-semipreinvex ($r$-semipreincave) when $r > 0$, and $r$-semipreincave ($r$-semipreinvex) when $r < 0$.*
(e) *Let $f : S \subset R^n \to R$ be $r$-semipreinvex ($r$-semipreincave) function with respect to $\eta$. Then $f$ is $s$-semipreinvex ($s$-semipreincave) function with respect to $\eta$ for any $s > r (s < r)$.*

*The following results characterize the class of $r$-semipreinvex functions.*

**Theorem 3.2** *Let $S \subseteq R^n$ be semi-connected set with respect to $\eta$, $f : S \to R$ is $r$-semipreinvex functions with respect to the same $\eta$ if and only if, for all $x, y \in S, \lambda \in [0, 1]$, and $u, v \in R$,*

$$\begin{cases} f(x) < u \text{ and } f(y) < v \Rightarrow f(y + \lambda\eta(x, y, \lambda)) < \log(\lambda e^{ru} + (1 - \lambda)e^{rv})^{\frac{1}{r}}, & r \neq 0, \\ f(x) < u \text{ and } f(y) < v \Rightarrow f(y + \lambda\eta(x, y, \lambda)) < \lambda u + (1 - \lambda)v, & r = 0. \end{cases} \tag{3}$$

*Proof* When $r = 0$, please refer to the proof of Theorem 2.1 of [24].
Let $r > 0$ (the proof in the case when $r < 0$ is analogous), and $f$ be $r$-semipreinvex functions with respect to $\eta$, and $f(x) < u$, $f(y) < v$, $0 < \lambda < 1$. From (2), it follows that

$$\begin{aligned} f(y + \lambda\eta(x, y, \lambda)) &\leq \log\left(\lambda e^{rf(x)} + (1 - \lambda)e^{rf(y)}\right)^{\frac{1}{r}} \\ &< \log\left(\lambda e^{ru} + (1 - \lambda)e^{rv}\right)^{\frac{1}{r}}. \end{aligned}$$

Conversely, let $x, y \in S, \lambda \in [0, 1]$. For any $\delta > 0$, $f(x) < f(x) + \delta$, $f(y) < f(y) + \delta$. From (3), we can get that, for $0 < \lambda < 1$,

$$\begin{aligned} f(y + \lambda\eta(x, y, \lambda)) &< \log\left(\lambda e^{r(f(x)+\delta)} + (1-\lambda)e^{r(f(y)+\delta)}\right)^{\frac{1}{r}} \\ &= \log\left(e^{r\delta}\lambda e^{rf(x)} + e^{r\delta}e^{rf(y)} - \lambda e^{r\delta}e^{rf(y)}\right)^{\frac{1}{r}} \\ &= \log\left((e^{r\delta})^{\frac{1}{r}}\left(\lambda e^{rf(x)} + (1-\lambda)e^{rf(y)}\right)^{\frac{1}{r}}\right) \\ &= \log\left(\lambda e^{rf(x)} + (1-\lambda)e^{rf(y)}\right)^{\frac{1}{r}} + \delta. \end{aligned}$$

Let $\delta \to 0$, it follows that $f(y + \lambda\eta(x, y, \lambda)) \le \log(\lambda e^{ru} + (1-\lambda)e^{rv})^{\frac{1}{r}}$. □

**Theorem 3.3** *Let $S \subseteq R^n$ be semi-connected set with respect to $\eta$. $f : S \to R$ is $r$-semi-preinvex functions with respect to the same $\eta$ if and only if the set*

$$F(f) = \{(x, u) : x \in S, u \in R, f(x) < u\}$$

*is semi-connected with respect to $\eta' : F(f) \times F(f) \times [0, 1] \to R^{n+1}$, where*

$$\begin{cases} \eta'((y, v), (x, u), \lambda) = \left(\eta(y, x, \lambda), \dfrac{\log\left(\lambda e^{r(v-u)} + (1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right), & r \neq 0, 0 < \lambda \le 1, \\ \eta'((y, v), (x, u), \lambda) = \eta(y, x, 0), & r \neq 0, \lambda = 0, \\ \eta'((y, v), (x, u), \lambda) = (\eta(y, x, \lambda), v - u), & r = 0. \end{cases}$$

*for all $(x, u), (y, v) \in F(f)$.*

*Proof* When $r = 0$, the proof was given by Yang et al. (see [24], Theorem 2.2).

When $r \neq 0, \lambda = 0$, the Theorem is obvious.

Let $r > 0, 0 < \lambda \le 1$ (the proof in the case when $r < 0$ is analogous). Let $(x, u) \in F(f)$ and $(y, v) \in F(f)$, i.e., $f(x) < u$ and $f(y) < v$. From (2), we have

$$\begin{aligned} f(y + \lambda\eta(x, y, \lambda)) &\le \log\left(\lambda e^{rf(x)} + (1-\lambda)e^{rf(y)}\right)^{\frac{1}{r}} \\ &< \log\left(\lambda e^{ru} + (1-\lambda)e^{rv}\right)^{\frac{1}{r}}. \quad \lambda \in (0, 1) \end{aligned}$$

Thus

$$\left(x + \lambda\eta(y, x, \lambda), \log\left(\lambda e^{rv} + (1-\lambda)e^{ru}\right)^{\frac{1}{r}}\right) \in F(f). \quad \lambda \in (0, 1)$$

That is,

$$\left(x + \lambda\eta(y, x, \lambda), u + \lambda\frac{\log\left(\lambda e^{r(v-u)} + (1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right) \in F(f). \quad \lambda \in (0, 1)$$

Then

$$(x, u) + \lambda\left(\eta(y, x, \lambda), \frac{\log\left(\lambda e^{r(v-u)} + (1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right) \in F(f). \quad \lambda \in (0, 1)$$

Hence, $F(f)$ is a semi-connected set with respect to

$$\eta'((y,v),(x,u),\lambda)=\left(\eta(y,x,\lambda),\ \frac{\log\left(\lambda e^{r(v-u)}+(1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right).$$

Conversely, assume that $F(f)$ is a semi-connected set with respect to $\eta'$. Let $x, y \in S$ and $u, v \in R$ such that $f(x) < u$, $f(y) < v$. Then, $(x,u) \in F(f)$ and $(y,v) \in F(f)$. From the semi-connectedness of the set $F(f)$ with respect to $\eta'$, we have

$$(x,u)+\lambda\eta'((y,v),(x,u),\lambda)\in F(f).\quad \lambda\in(0,1)$$

It follows that

$$(x,u)+\lambda\left(\eta(y,x,\lambda),\ \frac{\log\left(\lambda e^{r(v-u)}+(1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right)\in F(f).\quad \lambda\in(0,1).$$

Then

$$\left(x+\lambda\eta(y,x,\lambda),\log\left(\lambda e^{rv}+(1-\lambda)e^{ru}\right)^{\frac{1}{r}}\right)\in F(f).\quad \lambda\in(0,1)$$

that is

$$f\left(y+\lambda\eta(x,y,\lambda)\right)<\log\left(\lambda e^{ru}+(1-\lambda)e^{rv}\right)^{\frac{1}{r}}.$$

Thus, by Theorem 3.2, $f$ is $r$-semipreinvex function with respect to $\eta$ on $S$. □

**Theorem 3.4** *Let $S \subseteq R^n$ be semi-connected set with respect to $\eta$, A function $f : S \to R$ is $r$-semipreinvex functions with respect to the same $\eta$ if and only if the set*

$$G(f)=\{(x,u): x\in S, u\in R, f(x)\le u\}$$

*is semi-connected with respect to $\eta' : G(f)\times G(f)\times[0,1]\to R^{n+1}$, where*

$$\begin{cases}\eta'((y,v),(x,u),\lambda)=\left(\eta(y,x,\lambda),\ \frac{\log\left(\lambda e^{r(v-u)}+(1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right), & r\neq 0, 0<\lambda\le 1,\\ \eta'((y,v),(x,u),\lambda)=\eta(y,x,0), & r\neq 0, \lambda=0,\\ \eta'((y,v),(x,u),\lambda)=(\eta(y,x,\lambda), v-u), & r=0.\end{cases}$$

*for all $(x,u),(y,v)\in G(f)$.*

*Proof* When $r = 0$, the proof was given by Yang et al. (see [24], Theorem 2.3).

When $r \neq 0$, the proof of the Theorem is analogous with the Theorem 3.3 and is omitted. □

**Theorem 3.5** *Let $S \subset R^{n+1}$ and*

$$f(x)=\inf\{u: u\in R, (x,u)\in S\},\quad \forall x\in R^n. \tag{4}$$

*If $S$ is a semi-connected set with respect to $\eta' : S\times S\times[0,1]\to R^{n+1}$ :*

$$\begin{cases}\eta'((y,v),(x,u),\lambda)=\left(\eta(y,x,\lambda),\ \frac{\log\left(\lambda e^{r(v-u)}+(1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right), & r\neq 0, 0<\lambda\le 1\\ \eta'((y,v),(x,u),\lambda)=\eta(y,x,0), & r\neq 0, \lambda=0\\ \eta'((y,v),(x,u),\lambda)=(\eta(y,x,\lambda), v-u), & r=0.\end{cases}$$

*for all* $(x, u), (y, v) \in S$. *Among them,* $\eta : R^n \times R^n \times [0, 1] \to R^n$. *Then* $f : R^n \to R$ *is a* $r$*-semipreinvex function with respect to the* $\eta$.

*Proof* When $r = 0$, the proof was given by Yang et al. (see [24], Theorem 2.4)

When $r \neq 0, \lambda = 0$, the Theorem is obvious.

Let $r > 0, 0 < \lambda \leq 1$ (the proof in the case when $r < 0$ is analogous), and let $x, y \in R^n$. Since $S$ is a semi-connected set with respect to$\eta'((y, v), (x, u), \lambda)$, then, for any $(x, u), (y, v) \in S$,

$$(x, u) + \lambda\eta'((y, v), (x, u), \lambda) \in S, \quad \forall\lambda \in (0, 1).$$

From

$$\eta'((y, v), (x, u), \lambda) = \left(\eta(y, x, \lambda), \frac{\log\left(\lambda e^{r(v-u)} + (1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right),$$

It follows that

$$\begin{aligned}(x, u) &+ \lambda\eta'((y, v), (x, u), \lambda)\\ &= \left(x + \lambda\eta(y, x, \lambda), \log\left(\lambda e^{rf(y)} + (1-\lambda)e^{rf(x)}\right)^{\frac{1}{r}}\right) \in S, \quad \lambda \in (0, 1).\end{aligned}$$

From (4),

$$f\,(x + \lambda\eta(y, x, \lambda)) \leq \log\left(\lambda e^{rf(y)} + (1-\lambda)e^{rf(x)}\right)^{\frac{1}{r}}, \quad \forall\lambda \in (0, 1).$$

Hence, $f$ is a $r$-semipreinvex function with respect to $\eta$ on $R^n$. □

**Theorem 3.6** *Let* $I$ *be an index set.* $S \subseteq R^n$ *be semi-connected set with respect to* $\eta : R^n \times R^n \times [0, 1] \to R^n$, *and a family of real-valued functions* $(f_i)_{i\in I}$ *be* $r$*-semipreinvex with respect to the same* $\eta$ *and bounded from above on* $S$. *Then, the function* $f(x) = \sup_{i\in I} f_i(x)$ *is a* $r$*-semipreinvex function with respect to the same* $\eta$ *on* $S$.

*Proof* When $r = 0$, the proof was given by Yang et al. (see [24], Theorem 2.4)

Let $r \neq 0, 0 < \lambda \leq 1$. From the $r$-semipreinvexity with respect to the same $\eta$ of every $f_i$ and by Theorem 3.4, we can obtain that epigraph

$$G(f_i) = \{(x, u) | x \in S, u \in R, f_i(x) \leq u\}$$

is semi-connected set in $R^n \times R$ with respect to

$$\eta'((y, v), (x, u), \lambda) = \left(\eta(y, x, \lambda), \frac{\log\left(\lambda e^{r(v-u)} + (1-\lambda)\right)^{\frac{1}{r}}}{\lambda}\right).$$

Therefore, their intersection

$$\begin{aligned}\bigcap_{i\in I} G(f_i) &= \{(x, u) | x \in S, u \in R, f_i(x) \leq u, i \in I\}\\ &= \{(x, u) | x \in S, u \in R, f(x) \leq u\}\end{aligned}$$

is also a semi-connected set in $R^n \times R$ with respect to $\eta'$. By Lemma 2.1, the intersection is the epigraph of $f$. Again by Theorem 3.4, $f$ is a $r$-semipreinvex with respect to $\eta$. □

## 4 *r*-Semipreinvexity and optimality

In this section, we consider nonlinear programming problems without constraint and with constraint and obtain some optimality results under $r$-semipreinvexity. These results are similar to the results in [17]. Because $r$-semipreinvexity is generalization of $r$-preinvexity. Our results is generalization of corresponding results in [17].

In the following, we firstly give the optimality condition of nonlinear programming without constraint under $r$-semipreinvexity condition.

**Theorem 4.1** *Let $S \subset R^n$ be a semi-connected set with respect to $\eta$, $f : S \subset R^n \to R$ be a $r$-semipreinvex with respect to the same $\eta$. Then each local minimum of the function $f$ is its global minimum.*

*Proof* Assume that $f$ attains its local minimum at a point $y \in S$. We proceed by contradiction. Suppose that $y$ is not a global minimum of the function $f$ on $S$, i.e., there exists a point $x \in S$ such that

$$f(x) < f(y). \tag{5}$$

Let $f$ be $r$-semipreinvex with respect to $\eta$ on $R^n$ with $r > 0$ (the proof in the case when $r < 0$ is analogous; only the directions of some inequalities should be changed to the opposite ones). Then, for all $x, y \in S$, and any $\lambda \in (0, 1)$, From (2), we have

$$f\left(y + \lambda\eta(x, y, \lambda)\right) \leq \log\left(\lambda e^{rf(x)} + (1 - \lambda)e^{rf(y)}\right)^{\frac{1}{r}}.$$

That is

$$f\left(y + \lambda\eta(x, y, \lambda)\right) \leq \frac{1}{r}\log\left(\lambda e^{rf(x)} + (1 - \lambda)e^{rf(y)}\right). \tag{6}$$

Multiplying both sides of the inequality (6) by $r$ and since the exponential function is increasing, we get the following inequality

$$e^{rf(y+\lambda\eta(x,y,\lambda))} \leq \lambda e^{rf(x)} + (1 - \lambda)e^{rf(y)}$$

holds for any $\lambda \in (0, 1)$. Hence, and from (5), we get the inequality

$$e^{rf(y+\lambda\eta(x,y,\lambda)} - e^{rf(y)} \leq \lambda\left(e^{rf(x)} - e^{rf(y)}\right) < 0$$

holds for any $\lambda \in (0, 1)$. Thus, we get that inequality $f(y + \lambda\eta(x, y, \lambda) < f(x)$ which, for sufficiently $\lambda$ close to zero, contradicts the definition of a local minimum of $f$ at $y$.
In the case when $r = 0$ , the proof is analogous. □

Next, we consider the nonlinear programming with inequality constraint.

$$(P) \qquad \min f(x)$$
$$\text{s.t } g_i(x) \leq 0, i = 1, 2, \ldots, k, x \in S.$$

where $S$ is a nonempty subset of $R^n$, $f, g_i : S \to R, i = 1, 2, \ldots, k$. Denote the set of all feasible solutions for $(P)$ by

$$K := \{x \in S, g_i(x) \leq 0, i = 1, 2, \ldots, k\}.$$

**Theorem 4.2** *Suppose that $g_i (i = 1, 2, \ldots, k)$ is $r$-semipreinvex with respect to $\eta$ on $K$. Then the set of all feasible solutions $K$ is semi-connected with respect to the same $\eta$.*

*Proof* We prove only the case when $r \neq 0$. Let $x, y \in K$. We have

$$g_i(x) \leq 0, \quad g_i(y) \leq 0, i = 1, 2, \ldots, k. \tag{7}$$

By the $r$-semipreinvexity of $g_i(x)(i = 1, 2, \ldots, k)$ on $K$ and (7), we obtain

$$g_i(y + \lambda\eta(x, y, \lambda) \leq \log\left(\lambda e^{rg_i(x)} + (1-\lambda)e^{rg_i(y)}\right)^{\frac{1}{r}} \leq \log\left(\lambda + (1-\lambda)\right)^{\frac{1}{r}} = 0$$

holds for all $\lambda \in [0, 1]$, $i = 1, 2, \ldots, k$. Thus, $y + \lambda\eta(x, y, \lambda) \in K$. This shows that the set $K$ is semi-connected with respect to $\eta$. □

**Theorem 4.3** *Let $f$ and $g_i(i = 1, 2, \ldots, k)$ be $r$-semipreinvex function with respect to same $\eta$ on $K$, and let $\bar{x}$ be a local minimum in (P). Then $\bar{x}$ be a global minimum in (P).*

*Proof* The result can be obtained from the Theorems 4.1 and 4.2. □

**Theorem 4.4** *Let $\bar{x}$ be a global minimum in (P) and $\eta : K \times K \times [0, 1] \to R^n$ be a vector-valued function. $\eta(x, y, \lambda) \neq 0$, for all $x, y \in K, x \neq y$. If $f$ is strictly $r$-semipreinvex function on $K$ with respect to $\eta$ and $g_i(x)(i = 1, 2, \ldots k)$ is $r$-semipreinvex function on $K$ with respect to same $\eta$. Then $\bar{x}$ is the unique optimal solution in (P).*

*Proof* By the Theorem 4.2, we know that the set of feasible solutions $K$ is semi-connected set with respect to $\eta$. By contradiction. Let $\hat{x} \neq \bar{x}$ be optimal solution in ($P$), then $\hat{x} \in K$ and

$$f(\hat{x}) = f(\bar{x}). \tag{8}$$

Since $K$ is semi-connected set with respect to $\eta$, we have

$$\bar{x} + \lambda\eta\left(\hat{x}, \bar{x}, \lambda\right) \in K.$$

From the strictly $r$-semipreinvexity of $f$ with respect to $\eta$ on $K$ and (8), it follows that

$$\begin{aligned} f\left(\bar{x} + \lambda\eta(\hat{x}, \bar{x}, \lambda)\right) &< \log\left(\lambda e^{rf(\hat{x})} + (1-\lambda)e^{rf(\bar{x})}\right)^{\frac{1}{r}} \\ &= \log\left(\lambda e^{rf(\bar{x})} + (1-\lambda)e^{rf(\bar{x})}\right)^{\frac{1}{r}} \\ &= f(\bar{x}). \end{aligned}$$

holds for all $\lambda \in (0, 1)$. Then it means that $\bar{x}$ is not a global minimum in ($P$), contradicting the assumption. □

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